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Perverse coherent sheaves on a blowup surface

§ Introduction

As perverse coherent sheaves are closely related to wall-crossing, I start with it in the ordinary setting.

X : nonsingular projective surface $/\mathbb{C}$

H : ample line bundle

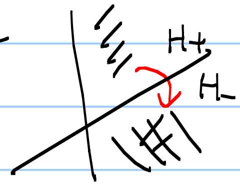
$M_H(v)$: moduli space of H-semistable sheaves E with $ch(E) = v$ $(v \in H^*(X, \mathbb{Q}))$

- Wall-crossing :

How moduli spaces change when H is moved ?

$\to$ Chamber structure on the ample cone

$\begin{cases} \text{unchanged on the chamber} \\ \text{changed if cross the wall} \end{cases}$



E_+ : H_+-stable, but not H_--stable

$\exists\, S$: subsheaf $\subset E_+$

$0 \to S \to E_+ \to Q \to 0$

$$\frac{\chi(S(mH_+))}{\operatorname{rk} S} \underset{<}{>} \frac{\chi(E(mH_+))}{\operatorname{rk} E} \underset{<}{>} \frac{\chi(Q(mH_+))}{\operatorname{rk} Q}$$

But $0 \to Q \to E_- \to S \to 0$ is H_--stable.

This is, more or less, what happens
in the wall-crossing.

But people gradually have realised that
there is a "larger" parameter space for
the stability condition.

Douglas (phys.), Bridgeland (math.)
$\mathrm{Stab}(X)$: cpx mfd of
dim. $= \mathrm{rk}\, K_{top}(X)$
$= \mathrm{rk}\, H_*(X)$

<u>Motivation 1</u>
Study moduli spaces for the larger param. space
in detail (in a specific example)

<u>Motivation 2</u>
Instanton counting
$\longleftarrow$ not mention today.

X : nonsingular projective surface $/\mathbb{C}$

$x \in X$ H : ample line b'dle

$$\begin{array}{ccc} p : \hat{X} & \longrightarrow & X \\ \cup & & \psi \\ C & \longrightarrow & p \end{array}$$

blowup at p

Def E : coherent sheaf on $\hat{X}$. is $\overbrace{\text{perverse}}^{\text{stable}}$ coherent

$\Longleftrightarrow$ (1) $\mathrm{Hom}(E, \mathcal{O}_C(-1)) = 0$

(2) $\mathrm{Hom}(\mathcal{O}_C, E) = 0$

(3) $p_* E$ is slope stable w.r.t. H

Example

$y \in C$ $0 \to \underset{\text{perverse}}{\mathcal{O}(-C)} \to \mathcal{I}_y \to \mathcal{O}_C(-1) \to 0$

But $\mathcal{I}_y$: not perverse as $\mathrm{Hom}(\mathcal{I}_y, \mathcal{O}_C(-1)) \neq 0$

exchange L, R $0 \to \mathcal{O}_C(-1) \to E \to \mathcal{O}(-C) \to 0$ $(\star)$

$\Longrightarrow$ E : perverse

Since $\dim \mathrm{Ext}^1(\mathcal{O}(-C), \mathcal{O}_C(-1)) = 1$, $\star$ is unique up to isom.

$= \mathrm{Hom}(\mathcal{O}(-C), \mathcal{O}_C(-1)[1])$

$\therefore$ Moduli of Perv. coh. "ideal" sheaves on $\hat{X}$ $\cong X$

Rem (1)~(3) $\Rightarrow$ $R^1 p_* E = 0$

$\therefore$ $\mathrm{ch}(p_* E)$: constant depending only on $\mathrm{ch}(E)$

$v \in H^*(\hat{X})$ $\quad$ ($p_* v$: (rank, deg) : coprime)

$$M^p_m(v) := \left\{ E \;\middle|\; \begin{array}{l} \mathrm{ch}\, E = v \\ E(-mC) : \text{perverse coherent} \end{array} \right\} \Big/ \text{isom.}$$

Th. Suppose $\langle v, [C] \rangle = 0$

$\Rightarrow$ ① $M^p_0(v) \cong M^X_H(p_* v)$ $\qquad$ ② $M^p_N(v)$ $\quad N = N(v) \gg 0$

$$\begin{array}{ccc} E & \longmapsto & p_* E \\ p^* F & \longleftarrow\!\shortmid & F \end{array} \qquad\qquad \cong M^{\hat{X}}_{p^*H - \varepsilon C}(v)$$

$$M^p_0 \swarrow \searrow M^p_1 \swarrow \searrow \cdots\cdots\cdots\cdots \quad M^p_N$$

moduli on X $\qquad\qquad\qquad\qquad$ moduli on $\hat{X}$

M^p_m & M^p_{m+1} are related by 'flip'

$$0 \to \mathcal{O}_C(-m-1)^{\oplus k} \to E_m \longrightarrow E' \to 0$$

$$\in \qquad\qquad\qquad {}^\tau M^p_m$$

$$0 \to E' \to E_{m+1} \longrightarrow \mathcal{O}_C(-m-1)^{\oplus k} \to 0$$

$$\begin{array}{ccc} M^p_m & & M^{p+1}_m \\ \cup & & \cup \\ \bigcup_{E'} \mathrm{Gr}(k, \mathrm{Ext}^1(E', \mathcal{O}_C(-m-1))) & \dashleftarrow\dashrightarrow & \mathrm{Gr}(k, \mathrm{Ext}^1(\mathcal{O}_C(-m-1), E')) \\ & \searrow \quad \swarrow & \\ & M^p_m \cap M^p_{m+1}(v - k\,\mathrm{ch}\,\mathcal{O}_C(-m-1)) & \end{array}$$

(k must be moved)

↳ This makes the change of Donaldson invariants very complicated.

Euler #'s $\quad M_0^p \quad M_1^p \quad \cdots \quad M_N^p$

rank $= 1$ $\quad \mathrm{Hilb}^n X \quad\quad \mathrm{Hilb}^n \hat{X}$

($M_0^p = \mathrm{Hilb}^n X$, $M_N^p = \mathrm{Hilb}^n \hat{X}$)

Take generating funct.

$$\sum_{n=0}^{\infty} e(M_m^p(n))\, q^n = ?$$

$$m=0 \quad \left(\prod_{d=1}^{\infty} \frac{1}{1-q^d}\right)^{e(X)} \qquad m=\infty \quad \left(\prod_{d=1}^{\infty} \frac{1}{1-q^d}\right)^{e(\hat{X})}$$

$e(\hat{X}) = e(X)+1$

(Göttsche)

Th. $\quad ? = \left(\prod_{d=1}^{\infty} \frac{1}{1-q^d}\right)^{e(X)} \times \prod_{d=1}^{m} \frac{1}{1-q^d}$

Rem. $m=1$ $\quad M_1^p(n) =$ nested Hilb. $\mathrm{Hilb}^{n-1,n}(X)$

$$= \{ (Z_1, Z_2) \mid Z_1 \subset Z_2, \ \mathrm{supp}\, Z_2 \setminus Z_1 = \{p\} \}$$

Recover Cheah's formula

For general m, $M_m^p(n)$ is again an incidence variety.

Higher rk case.

$$\sum_{v} e(M_m^P(v))\, q^{v.\dim/2r} \Big/ \sum_{v} e(M_H^\times(p_* v))\, q^{v.\dim/2r}$$

$$= \sum_{\substack{k_1, \dots, k_r \in \mathbb{Z} \\ k_1 + \dots + k_r = \langle v, [C] \rangle \\ k_\alpha \geq -m}} \prod_{d=1}^{m+k_\alpha} \frac{1}{1-q^d} \times q^{(\vec{k}, \vec{k})/2}$$

↖ finite sum

$$\xrightarrow[m \to \infty]{} \Theta_{\mathbb{Z}^{r-1}+\alpha}(q) \Big/ \prod_{d=1}^{\infty} (1-q^d)^r$$

$\vdots$

Θ-func.

(proved earlier by Yoshioka

How we find the definition?

ⓐ Bridgeland's paper

ⓑ King's description via quiver

ⓐ $\mathcal{A} = D(\hat{X})$: derived category of coherent sheaves on $\hat{X}$

$\mathcal{B} = D(X) \xrightarrow{Lp^*} D(\hat{X})$ full subcategory

$\mathcal{C} = \mathcal{B}^{\perp} = \{ a \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(b, a) = 0 \quad \forall b \in \mathcal{B} \}$

$= \{ a \in \mathcal{A} \mid Rp_*(a) = 0 \}$

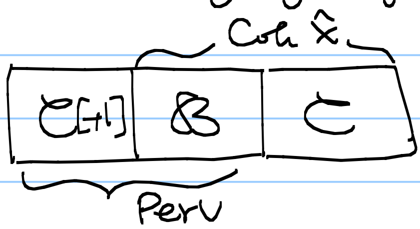
$= \{$ direct sums of $\mathcal{O}_C(-1)[m] \}$ (Orlov)

$\mathcal{A} = \langle \mathcal{C}, \mathcal{B} \rangle$: torsion pair

"torsion" "torsion-free"

$\Rightarrow \mathrm{Perv}(\hat{X}/X) \subset D(\hat{X})$ perverse coherent sheaf

obtained by "gluing" $(\mathcal{B}$ & $\mathcal{C}) \cap \mathrm{Coh}\,\hat{X}$ in a different way



$H^{-1}(E) \in \mathcal{C} \cap \mathrm{Coh}\,\hat{X}$

$H^{0}(E) \in \mathcal{B} \cap \mathrm{Coh}\,\hat{X}$

Bridgeland considered more general setting

- birational
- $Rp_*(\mathcal{O}_{\hat{X}}) = \mathcal{O}_X$
- relative dim. one

Def. (Bridgeland)

$E \in D(\hat{X})$ is perverse coherent $(\in \mathrm{Perv}(\hat{X}/X))$

(i) $H^i(E) = 0$ for $i \neq -1, 0$

(ii) $p_*(H^{-1}(E)) = 0$, $R^1 p_*(H^0(E)) = 0$

(iii) $\mathrm{Hom}(H^0(E), c) = 0 \quad \forall c \in \mathcal{C} \cap \mathrm{Coh}\,\hat{X}$

$H^0(E) \in \mathcal{B} \cap \mathrm{Coh}\,\hat{X}$

- $Rp_* E \in \mathrm{Coh}\,X$ Thus $\mathrm{Perv}(\hat{X}/X)$ is close to $\mathrm{Coh}\,X$.
- $\mathrm{Perv}(\hat{X}/X)$ is an abelian category.

Rem. Bridgeland considered 3-dim'l situation

- Moduli of Perv. coh "ideal" sheaves $\cong X^+$ flop
- $D(X) \cong D(X^+)$ given by FM transform w.r.t. the universal family

(b) (E, Φ) : framed $\overset{\text{torsion free}}{\text{sheaf}}$ on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$

$$\Phi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$$

framed moduli. $\cong \left\{ \begin{array}{c} B_1 \circlearrowright V \circlearrowleft B_2 \\ j \downarrow \uparrow i \\ W \end{array} \;\; \begin{array}{l} a)\ [B_1, B_2] + ij = 0 \\ b)\ S \subset V \text{ s.t. } \\ \quad \mathrm{Im}\, i \subset S,\ B_\alpha(S) \subset S \end{array} \right\} / GL(V)$

where V, W : vector spaces of $\dim = c_2(E)$, $\operatorname{rank} E$

○ variant on $\hat{\mathbb{P}}^2$ (after King)

$$\begin{tikzcd} V_0 \arrow[r, "d"] & V_1 \arrow[l, shift right, "B_2"'] \arrow[l, shift right=2, "B_1"'] \arrow[dl, "j"] \\ W \arrow[u, "i"] & \end{tikzcd}$$

$B_1 d B_2 - B_2 d B_1 + ij = 0$

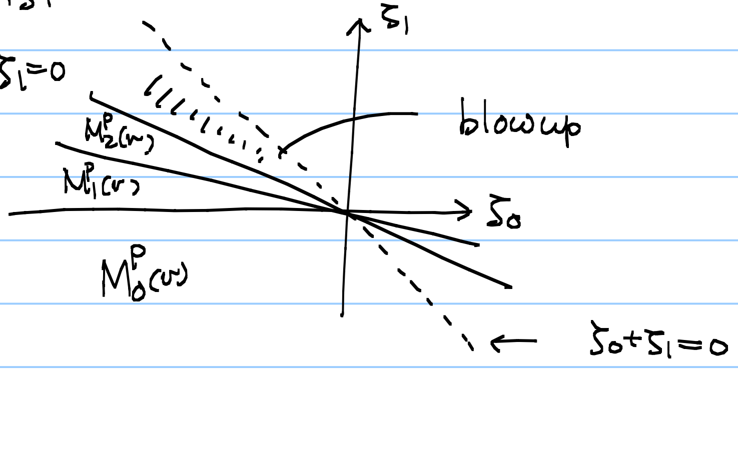
$\left(\begin{array}{l} \dim V_0 = \dim V_1 = c_2 \\ \dim W = \operatorname{rank} \end{array} \right.$

Taking GIT quot. w.r.t. the trivial line b'dle with nontrivial action: $GL(V_0) \times GL(V_1) \xrightarrow{\chi} \mathbb{C}^*$

$$\chi = \chi_{\zeta_0, \zeta_1} = (\det g_0)^{\zeta_0} \cdot (\det g_1)^{\zeta_1}$$

$m\zeta_0 + (m+1)\zeta_1 = 0$

$m \in \mathbb{Z}_{\geq 0}$



We can construct the moduli for param. on the wall.

<u>Rem</u> King considered the moduli space
for $\zeta_0 + \zeta_1 = 0$

— framed moduli space of loc. free sheaves

— " instantons

Hitchin-Kobayashi